Synopsis of the Proposed Ph.D. Program in Mathematics

# STUDY ON COEFFICIENT BOUNDS OF CERTAIN UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

Submitted to

**SWAMI RAMANAND TEERTH MARATHWADA UNIVERSITY**

Nanded-431606, Maharashtra State, India.

Submitted by

**K. C DESHMUKH**

Department of Mathematics

BahirjiSmarakMahavidyalay, Basmatnagar,

Maharashtra – 431512.

e-mail: kishord738@gmail.com

Under the Guidance of

**Dr. RAJKUMAR N. INGLE**

Head & Associate Professor

Department of Mathematics

BahirjiSmarakMahavidyalay, Basmatnagar,

Maharashta-431512

e-mail: ingleraju11@gmail.com

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## Abstract

In recent years, the properties and characteristics of many different subclasses of univalent multivalent functions have been systematically investigated, making it an important part. Various researchers have studied the properties of functions in the class S and its subclasses such as Starlike, Strongly Starlike, Convex, Close to Convex, and so on over the last three decades. In the opposite direction, a new subclass of functions involving operators was introduced. Differential operators are powerful mathematical tools for discovering new subclasses of S. Salagean Operator is a well-known differential operator. The purpose of this research is to look into the properties of convolution and subordination. Some growth and distortion theorems must be obtained. In order to obtain coefficient estimates for the functions of these classes, we need a research investigation certain inclusion relations, as well as their applications, will be obtained. The effect of the Noor integral operator and the Srivastava-Attiya operator on some well-known subclasses of the class S was investigated. The method of subordination is primarily used to study inclusion relations. Some examples of how these inclusion relations can be used will be provided. As a result of the research to be conducted, this research work will demonstrate that these classes are invariant under Bernardi's Integral Operator, subject to certain parameter conditions.

## Introduction

The concept of univalent as well as multivalent functions play significant role in geometric functions theory and have a new perspective on their own merits to stand as a specialization as well as to pursue the research work on this direction. The origin of the study trace back to a paper of Koebe (1907) on the uniformization of algebraic curves. In that paper, Koebe [1] proved that there is a constant k such that the boundary of the map of unit disc by any function in S (the class of univalent functions) is always at a distance not less than k from origin. Bieberbach [2] rediscovered that value of the constant k = 1/4 and also proved that $a_2 \leq 2$ for every $f \in$ S. The conjecture determined the course of research in univalent functions theory nearly seventy- year and was proved by de Branges [3] in 1985. During the period when the Bieberbach [2] conjecture remained unsolved, it was verified for several subfamilies of S. These sub-families have many interesting properties which make their study important in their own right. Many new techniques have been developed and introduced to study univalent and multivalent functions in broader perspective. A complex-valued function is said to be analytic in a domain D (a non-empty open connected subset of the complex plane C) if it has a uniquely determined derivative at each point of D.

**Definition 1.** A function *f*(z) is said to be univalent in D if for any two points

$$z_1, z_2 \in D, \quad z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2).$$

A necessary condition for an analytic function $f(z)$ to be univalent in D is that $f'(z) \neq 0$ in D. That, this condition is not sufficient, can be seen by considering the function $f'(z) \neq 0$ whose derivative never vanishes but clearly, it is not univalent in C. Regular and univalent functions have the nice property of mapping simply connected domains conformally onto simply connected domains. Because the Riemann mapping allows one to map any simply connected domain (other than the entire complex plane) in a oneone conformal m anner on the open unit disc, one can confine, without loss of generally, to functions univalent and analytic in the open unit disc z: |z| < 1. Throughout this work, the open unit disc will be denoted by E.If $g(z)$ is analytic and univalent in the unit disc E, so is the function $f(z) = \dfrac{g(z) - g(0)}{g'(0)}$ since $g'(0) \neq 0$ .Thus it is enough to consider functions in E satisfying $f(0) = 0, f'(0) = 1$.. Let A denote the class of functions $f$ analytic in E and normalized by the conditions $f(0) = 0, f'(0) = 1$.

Because the theory of univalent functions is well known, the goal of this work is to provide a brief overview of some univalent function subclasses. A function $f(z)$ in S is said to be Starlike with respect to the origin or simply Starlike in E if and only if the domain *f*(E) is Starlike with respect to the origin i.e. the line segment joining w=0 to any point of *f*(E) lies wholly in *f*(E). We shall denote by ST. the subclass of S consisting of Starlike functions. A function $f(z)$ in S is said to be convex in E if *f*(E) is a convex domain. We shall denote by CV the class of all members of S which are convex in E. Clearly CV $\subset$ ST the containment is proper since Koebe function $z/(1-z)^2$ is in ST but not in CV.The classical Alexander result that $f \in$ CV if and only if z $f' \in$ ST does not hold between the classes UCV and UST. Ronning [4] defined a subclass of Starlike functions Sp with the property that a function $f \in$ UCV if and only if z $f' \in$ Sp.In this study, we will attempt to introduce and investigate new subclasses of the normalized univalent analytic function S in the open unit disc, as well as consider the properties of the classes. Furthermore, we will investigate the radii of Starlikeness and Convexity for the classes that will be introduced.Using a specific multiplier transformation, I will investigate some more subclasses of univalent analytic functions. These functions extend the concept of uniformly close-to-convexity by mapping the open unit disc onto domains formed by parabolic and hyperbolic regions. Some interesting properties of these classes will be discussed, such as inclusion results, coefficient problems, distortion theorems, and invariance under certain integral operators.

I intend to look for new subclasses of meromorphically uniformly convex functions defined by the Bessel operator. Some intriguing properties of these classes are to be discovered. For a class of multiplier transformations associated with Noor integral operators defined on the space of normalized analytic functions in the open unit disc, I will obtain some subordination and superordination preserving properties. These transformations' sandwich-type theorems must be considered.

I'll try to introduce some new classes of Meromorphic functions in the punctured unit disc using the concept of k-uniformly bounded boundary rotation and the Rusheweyh differential operator. These classes will be investigated using the convolution technique and the principle of subordination. The inclusion results, generalized Bernardi integral operator, and coefficient growth rate will all be investigated. I'll look into using the Salagean differential operator to introduce new subclasses of univalent analytic functions. I'll derive the coefficient inequalities and other interesting properties for the functions in the classes I'll introduce here.

I'll try to create a new subclass of uniformly convex functions with negative coefficients defined by convolution . The main goal is to obtain coefficient estimates, distortion bounds, closure theorems, and extreme points for fu nctions in this new class. The results will be generalized to families with finitely many coefficients.

## Review of Literature and Development in the Subject

Koebe [1] pioneered the study of univalent functions in 1907, when he discovered that the range of all functions in S contains a common disc |w| < b, where b is an absolute constant. The Koebe function demonstrates that b ≤ ¼. Bieberbach [2] established that b can be taken to be 1/4, and he also demonstrated that if S, then equality occurs only for rotations of the Koebe function. Because the Koebe function appears to be an external function for so many problems in class S, Bieberbach [2] suspected that it maximises $|a_n|$ for every n, resulting in the famous Bieberbach conjecture, first proposed in 1916.

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, then $|a_n| \leq n$ , $for\ n = 2,3,\ldots$.Equality occurs for each $n$ for the Koebe function or one of its rotations.For many years, this famous problem served as a challenge, inspiring the development of important concepts and ingenious techniques that now serve as the foundation of the entire subject.

Bernardi [5], Hayman [6], and Goodman [7] have reviewed various developments in the theory of univalent fun ctions. Certain aspects of the subject have also been addressed in books by Nehari [8], Goluzin [9], and Jenkins [10]. Bernardi's [5] exhaustive Bibliography of schlicht functions contains over 4,200 references available in the literature on the theory of univalent functions up to 1981. Rusheweyh's book [11] provides an overview of all t he results in convolution theory for univalent functions. Robertson [12] introduces the concept of order of Starli keness as well as the concept of convex functions of order.Still another subclass of Starlike functions that has

occupied wide attention of workers in the field in recent years is the class of strongly Starlike functions introduced by Brannan and Kirwan [13] and Stankiewicz [14].

## Objectives of the Research

The main objective of the proposed research will be a contribution to geometric functions theory, more precisely to investigate the following aspects:

1. Applications of differential subordination techniques in the context of geometric functions theory.

2. Class preserving properties of some subclasses of univalent and multivalent functions. The notion of differential subordination for analytic functions generalizes the idea of differential inequality involving real functions. Using the concept of differential subordination an attempt shall be made to study the basic properties of certain subclasses of analytic univalent/multivalent functions. Using concepts of subordination which is well known in the literature, new classes of analytic functions shall be defined. Basic properties of these classes shall be studied. The properties to be investigated are: closure, inclusion, Hardamard product, growth, radius of convexity, order of starlikeness, integral transform, class preserving operators etc.

The purpose of this research is to review some certain subclasses of Univalent and multivalent functions and recently developments in this area. The broad aim of this study is critically examine the integral and differential operator and their inclusion properties. The main objective of this study is to provide some new properties of certain subclass of Univalent and multivalent functions and to study Inclusion relations of these subclasses.

## The study has the following objectives:

To understand the various subclasses of Univalent and multivalent functions such as Starlike, Convex, Close to Convex,Bessel functions, q-derivatives operator, Lucas polynomials, Borel distribution series, pascal and geganbaur polynomials etc.

To evaluate properties and characteristics of these subclasses.

To obtain partial sums and neighborhood properties of subclasses.

To find coefficients bounds of bi-univalent and multivalent functions.

To obtain coefficient estimates,distortion bounds for the functions of these classes.

To obtain certain inclusion relations and their applications.

To examine the applications of these subclasses.

## Research Methodology to be adopted

I will try to find new results and extend subclasses of Univalent multivalent functions using concepts, previously known theorems, lemmas, and logical methods. In my thesis, I will use various techniques to validate various subclass results. On the other hand, I will use the Convolution Method to demonstrate inequality of subclasses. With the help of a differential operator such as the Salagean operator, I will create new subclasses of S. It is hoped that our contribution to the title of this topic is completely unique and original. Several known results in the theory of Univalent Analytic function subclasses will be followed by new properties of certain subclasses obtained in the current research work. I will investigate the impact of the q-calculus Integral Operator, the SrivastavaAttiya Operator, Geganbaur polynomials, and the Borel distribution se ries on some wellknown subclasses of the class S. Inclusion relations for these classes are primarily studied usin g the case-by-case method of differential subordination.

**Importance of the Study**

1. **InternationalStatus**:

Koebe initiated international research into the theory of univalent functions in 1907. Analytic and univalent fun ctions have numerous subclasses.

Silverman's [15] introduction of a class T of functions with negative coefficients opened up a new and fruitful line of research in the theory of univalent functions.

In the investigation of a univalent function with negative coefficients there resulted in a new direction of research in this theory namely, the construction of subclasses of analytic univalent function is trying to settle a well-known conjecture known as the Bieberbach conjecture which was positively settled by Branges in 1985.

2. **National Status:**

T.G.Shabha [16] introduced a new subclass of p-valent functions with a negative coefficient defined by Hadamard product associated with a generalized differential operator. Radii of close-to-convexity,starlikeness, and convexity of the class are obtained. Also, distortion theorem, growth theorem, and coefficient inequalities are established.Vinod Kumar and Srivastsva [17] an extended fractional differintegral operator $\Omega_z^{(\lambda,p)}(-\infty < \lambda < p+1: p \in \mathbb{N})$, suitable for the study of multivalent functions is introduced. The various results obtained here for each of these function classes include coefficient bound, inclusion relation for $(k,\theta) -$ neighborhood of subclass of analytic and multivalent functions with negative coefficient, Hadamard products, Integral means. Further, results based on partial sums of functions belonging to the class are derived.

**3. Significance:**

The study of specific subclasses of Univalent Analytic functions is critical in the branch of Complex Analysis. The theory of Univalent Functions, one of the most fascinating subjects in Geometric Functions, necessitates th e development of specific subclasses.Differential Equations, Partial Differential Equations, Fractional Differenti al Equations, Meromorphic Functions, Harmonic Functions, Banach Spaces, and Functions of Several Complex Variables are just a few of the fields where theorems and applications of the current topic play an important role.

## Proposed Work Plan and Structure of the Study

**First Year Plan:**

In the first year I will collect the literature and survey on the subject related to the chosen topic by way of primary sources, secondary sources, research papers etc.,

I will study basic concepts and history of subclasses of Univalent Analytic functions.Theory of certain subclasses like Starlike, Convex, Close to Convex, etc.I will attend some international/national conferences and seminars to get myself acquainted deeply with the subject concerned in depth and also to get a broad understanding of the topic chosen from the Resource Persons and Presenters in the field of the topic entitled for my research work.

**Second Year Plan:**

I will study some new integral operators and their inclusion properties. I will study the effect of integral operators on some well-known subclasses and record them in the thesis accordingly.

I will refer references from conferences, seminars, workshops and internet and try to determine some new results. I will present Research Papers /Articles in repute Mathematical Journals and present papers in seminars and conferences on the selected topic and publication of the same accordingly.

**Third Year Plan:**

I will study the inclusion relations of certain subclasses and their applications. I will try to explore new subclasses and properties of Univalent Analytic functions.

I will try to elicit applications of these inclusion properties of Univalent Multivalent functions to other areas of mathematics.Submission of my completed research work to Swami RamanandTeerthMarathwada University, Nanded-431606, Maharashtra State, India for the Award of the Degree of Ph.D. in Mathematics.

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K. C.Deshmukh

Research Scholar

Dr. R. N. Ingle

Research Guide / Supervisor